

## THE COSMIC NO-HAIR CONJECTURE A STUDY OF THE NARIAI SOLUTIONS

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In this talk, we investigate the cosmic no-hair conjecture for perturbed Nariai solutions within the class of Gowdy symmetric solutions of Einstein's field equations in vacuum with a positive cosmological constant. In particular, we are interested whether these perturbations allow to construct new cosmological black hole solutions.

*Keywords:* Cosmology; inflation; cosmic no-hair; black holes; numerical relativity.

### Introduction and motivation

The successful interpretation of cosmological observations requires a deep and fundamental understanding of rapidly expanding solutions of Einstein's field equations beyond spatial homogeneity and isotropy. The so-called standard model of cosmology, which is based on these symmetry assumptions, is surprisingly consistent with current observations.<sup>1,2</sup> A keystone of the standard model is inflation, i.e. a period characterized by accelerated expansion. Since one can expect that strong inhomogeneities caused by quantum fluctuations and other physical processes are dominant in the very early universe, it is, however, important to study the inhomogeneous case. A particular concern is whether inflation is able to homogenize and isotropize *generic* initial inhomogeneities. If so, this would yield a natural explanation for the apparent homogeneity and isotropy of our present universe, without making ad-hoc assumptions on the initial conditions. The conjecture that generic expanding solutions of Einstein's field equations in vacuum with cosmological constant  $\Lambda > 0$ ,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \quad (1)$$

approach a particular homogeneous and isotropic solution asymptotically, namely the de-Sitter solution, is known as the cosmic no-hair conjecture.<sup>3,4</sup> In the following discussion, we use this conjecture in the form of Conjecture 1 in Ref. 5. Recall that  $\Lambda > 0$  in Einstein's field equations is the simplest model for dark energy which is consistent with all current observations; see the references above.

Although there is some support for this conjecture in special situations,<sup>6–11</sup> the general case remains unclear due to the complexity of Einstein's field equations. From this point of view, a particularly interesting family of solutions of Eq. (1) is the class of Nariai solutions.<sup>12,13</sup> These are *simple* solutions which are *not* consistent with the cosmic no-hair picture. The (standard) Nariai solution of Eq. (1) is

$$g = (-dt^2 + \cosh^2 t d\rho^2 + g_{\mathbb{S}^2})/\Lambda, \quad (2)$$

on the manifold  $M = \mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2)$  for any  $\Lambda > 0$ . Here,  $g_{\mathbb{S}^2} = d\theta^2 + \sin^2 \theta d\phi^2$  in standard polar coordinates  $(\theta, \phi)$  on  $\mathbb{S}^2$ . In Ref. 5, we also define *generalized* Nariai solutions whose properties are listed in the references above. In the following we often speak of *the* Nariai solution when we mean any generalized Nariai solution.

It is easy to see from Eq. (2) that the asymptotics for  $|t| \rightarrow \infty$  are particularly peculiar. While the  $\mathbb{S}^1$ -factor of the spatial slices expands exponentially for increasing positive  $t$ , the volume of the  $\mathbb{S}^2$ -factor stays constant. Thus, the expansion of this solution is anisotropic in the sense that the shear tensor of the  $t = \text{const}$ -slices never approaches zero. This implies that the cosmic no-hair picture does not hold with respect to the foliation of  $t = \text{const}$ -surfaces. This alone does not mean that the cosmic no-hair conjecture is false. On the one hand, it remains possible that there exist other foliations of the Nariai solutions, presumably based on non-symmetric surfaces, for which the cosmic no-hair picture is attained. However, in Ref. 5, we prove the following statement based on the results in Ref. 11: “The Nariai solutions do not have even a patch of a smooth conformal boundary. The same is true for their universal cover.” This suggests that the Nariai solutions violate the cosmic no-hair picture for any choice of foliation because, if a given foliation approached a homogeneous and isotropic foliation of the de-Sitter solution, it would presumably approach a patch of a smooth conformal boundary.

## Perturbations and cosmological black hole solutions

Hence, if the cosmic no-hair conjecture is true, the asymptotics of the Nariai solutions should be special and should be instable under perturbations. We expect that generic perturbations either collapse and form a singularity in a given time direction, or if there is expansion, they approach the de-Sitter solution. In general, when we speak of a perturbation of a Nariai solution in the following, we mean a cosmological solution of the fully non-linear Einstein’s field equations (1) whose data, on some Cauchy surface, is close to the data on a Cauchy surface of a given Nariai solution. In principle, we are interested in generic perturbations without symmetries. In practical investigations, however, we must make simplifying assumptions. In a first step in Ref. 5, we investigated the spatially homogeneous (but anisotropic) case of perturbations and proceeded with the Gowdy case<sup>14,15</sup> in Ref. 16.

Spatially homogeneous (but anisotropic) solutions with spatial  $\mathbb{S}^1 \times \mathbb{S}^2$ -topology of the vacuum field equations with  $\Lambda > 0$  are either Schwarzschild-de-Sitter or generalized Nariai solutions locally. This is derived in Ref. 5 and references therein, and yields a full description of the expected instability of the Nariai solutions with respect to spatially homogeneous perturbations as follows. It turns out that there is a parameter  $H_*^{(0)}$  introduced in Ref. 5, whose sign controls the instability. This quantity represents the initial value of the expansion of the spatial  $\mathbb{S}^2$ -factor with respect to a foliation of homogeneous surfaces. For  $H_*^{(0)} = 0$ , we get a generalized Nariai solution and the spatial  $\mathbb{S}^2$ -factor stays constant in time. If  $H_*^{(0)}$  is an arbitrarily small positive number and the direction of time is chosen such that the spatial  $\mathbb{S}^1$ -factor is initially expanding, then the expansion of the  $\mathbb{S}^2$ -factor increases during the future evolution and eventually, the foliation becomes consistent with the cosmic no-hair picture. In this case, the solution forms a smooth future conformal boundary. If, however,  $H_*^{(0)}$  is a negative number with arbitrarily small modulus, then the spatial  $\mathbb{S}^2$ -factor collapses and the solution forms cigar-type curvature singularity in the future.

It was the idea of Bousso<sup>17</sup> to exploit this instability of the Nariai solutions to construct inhomogeneous cosmological black hole solutions by making the parameter  $H_*^{(0)}$  spatially

dependent. The expectation is that the local value of its sign determines whether the solutions forms either black hole interior regions or cosmologically expanding regions. In his article he discusses the spherically symmetric case and is able to confirm the expectation using heuristic arguments. The resulting cosmological black holes are the Schwarzschild-de-Sitter solutions.<sup>18</sup> It was the aim of our discussion in Ref. 16 to study the same problem for Gowdy symmetry. The Gowdy case is more challenging and numerical techniques are necessary. In our paper, we derive a family Gowdy initial data on  $\mathbb{S}^1 \times \mathbb{S}^2$  close to certain generalized Nariai solutions and evolve them numerically so that the quantity  $H_*^{(0)}$  is spatially dependent.

To our surprise, the numerical results, obtained with a numerical code based on Ref. 19, contradict our expectations.<sup>16</sup> Namely, eventually the solutions “make a decision” whether the spatial  $\mathbb{S}^2$ -factor expands or collapses *globally* in space and the expected local behavior is suppressed. There appears to be a new critical solution, i.e. a critical value  $\mu_c$  of a parameter  $\mu$  so that for  $\mu < \mu_c$ , the solution collapses globally in space, and for  $\mu > \mu_c$ , expands globally in space. It would be interesting to identify the critical solution and to study whether critical phenomena, which play such an important role<sup>20</sup> for the critical collapse of black holes, also occur here. In summary, our results give evidence that it is not possible to construct cosmological black hole solutions by means of small Gowdy symmetric perturbations of the Nariai solutions. We hope to shed further light on this for instance by means of linearization. Moreover, we will study “large” perturbations, and our preliminary results suggest that in contrast to “small” perturbations those show the expected local behavior associated with cosmological black holes.

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